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ARTICLES

Partial Differential Equations

Symmetry reductions, exact solutions, and conservation laws of the generalized Zakharov equations
Eerduo Duhe and George W. Bluman
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generalized Zakharov equations

Eerdun Buhe and George W. Bluman

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In this paper, the generalized Zakharov equations, which describe interactions between high- and low-frequency waves in plasma physics are studied from the perspective of Lie symmetry analysis and conservation laws. Based on some subalgebras of symmetries, several reductions and numerous new exact solutions are obtained. All of these solutions represent modified traveling waves. The obtained solutions include expressions involving Airy functions, Bessel functions, Whittaker functions, and generalized hypergeometric functions. Previously unknown conservation laws are constructed for the generalized Zakharov equations using the direct method. Profiles are presented for some of these new solutions. © 2015 AIP Publishing LLC.
of Eqs. (1) describes nonlinear self-interaction in the high-frequency subsystem which corresponds to a self-focusing effect in plasma physics. The GZEs are a universal model of interaction between high- and low-frequency waves in one dimension. When \( \beta = 0 \), (1) reduces to the classical Zakharov equation of plasma physics\(^{19}\) and has an important particular solution — the Langmuir soliton. Obviously, the GZEs is a strongly nonlinear system and it is quite difficult to obtain its solitary wave solutions.

Recently, extensive studies have been carried out by many authors on the GZEs. In Refs. 20 and 21, the solitary wave solutions of the GZEs have been obtained by the well-known He’s variational approach. Wang and Li\(^{22}\) introduced periodic wave solutions of GZEs using the extended F-expansion method. In Ref. 23, the Painlevé property and some exact solutions of the GZEs have been obtained using the truncated Painlevé expansion. Dai and Xu\(^{24}\) studied the dynamic behaviors of some exact traveling wave solutions to the GZEs.

In order to study Lie symmetry reductions, exact solutions, and conservation laws of GZEs (1), we express the complex envelope as \( E(x,t) = u(x,t) + iv(x,t) \), with real high-frequency waves \( u(x,t) \) and \( v(x,t) \). After substituting the complex envelope expression into (1) and separating its real and imaginary parts, we obtain

\[
\begin{align*}
2uF - 2\beta u(u^2 + v^2) - v_t + u_{xx} &= 0, \\
2vF - 2\beta v(u^2 + v^2) + u_t + v_{xx} &= 0, \\
F_{tt} - F_{xx} + (u^2 + v^2)_{xx} &= 0.
\end{align*}
\]

(2)

The layout of the rest of this paper is as follows: In Section II, some exact solutions of (2) are obtained by using Lie symmetry reductions along with some algebraic analysis under some Lie optimal subalgebras. In Section III, conservation laws of (2) are constructed through use of the direct (multiplier) method. Finally, conclusions are summarized in Section IV.

II. LIE SYMMETRY ANALYSIS OF GZES

The point symmetry group of generalized Zakharov equations (2) is generated by a vector field of the form

\[
\Gamma = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x^i} + \sum_{j=1}^{3} \eta_j \frac{\partial}{\partial \theta^j},
\]

(3)

where the infinitesimals \( \xi_i = \xi_i(x^\alpha, \theta^\beta) \), \( \eta_j = \eta_j(x^\alpha, \theta^\beta) \), and \( (x^1, x^2) = (x,t) \), \( (\theta^1, \theta^2, \theta^3) = (u,v,F) \). The application of the second extension \( \Gamma^{(2)} \) to (2) results in an overdetermined system of linear partial differential equations for the infinitesimals.

Wu’s method (also called the characteristic-set algorithm)\(^{25–28}\) was established by the Chinese mathematician Wu Wen Tsun in the 1970s, based on Ritt’s theory. The differential analogue of Wu’s method was proposed in the 1980s.\(^{26}\) The method is especially on target to deal with the zero set of a differential polynomial system and efficient differential elimination without directly involving the concept of an algebra ideal. Here, the general solutions of the overdetermined system of linear partial differential equations with the aid of Wu’s method is given by

\[
\begin{align*}
\xi_1(x,t,u,v,F) &= C_1, \\
\xi_2(x,t,u,v,F) &= C_2, \\
\eta_1(x,t,u,v,F) &= -(C_3t^2 + 2C_4t - C_5)v, \\
\eta_2(x,t,u,v,F) &= (C_3t^2 + 2C_4t - C_5)u, \\
\eta_3(x,t,u,v,F) &= C_3t + C_4,
\end{align*}
\]

(4)

where \( C_i, i = 1, \ldots, 5 \) are five arbitrary constants. Hence, the point symmetries of (2) form a five dimensional Lie algebra \( \mathfrak{g} \) spanned by the following linearly independent operators:
\[ \Gamma_1 = \frac{\partial}{\partial t}, \]
\[ \Gamma_2 = \frac{\partial}{\partial x}, \]
\[ \Gamma_3 = -t^2v \frac{\partial}{\partial u} + t^2u \frac{\partial}{\partial v} + t \frac{\partial}{\partial F}, \]
\[ \Gamma_4 = -2tv \frac{\partial}{\partial u} + 2tu \frac{\partial}{\partial v} + \frac{\partial}{\partial F}, \]
\[ \Gamma_5 = v \frac{\partial}{\partial u} - u \frac{\partial}{\partial v}. \]

The operators \( \Gamma_1 \) and \( \Gamma_2 \) are related to the obvious invariance of Eqs. (2) under space and time translations, respectively. The point symmetry vector field \( \Gamma_5 \) is related to invariance of Eqs. (2) under rotations in \( u - v \) space.

### A. One-dimensional optimal system of subalgebras

One does not need to use all the one-dimensional subalgebras of Lie algebra (5) to construct invariant solutions. However, a well-known standard procedure allows one to classify all one-dimensional subalgebras into subsets of conjugate subalgebras. This involves constructing the adjoint representation group, which introduces a conjugate relation in the set of all one-dimensional subalgebras. If one uses only one representative from each family of equivalent subalgebras, an optimal set of subalgebras is created. The corresponding set of invariant solutions is then the minimal list from which one can obtain all other invariant solutions of one-dimensional subalgebras simply via transformations. In this subsection, we present the optimal system of one-dimensional subalgebras for the Lie algebra for point symmetries (5) of system (2) to obtain the corresponding optimal set of group-invariant solutions.

Each \( \Gamma_i \) of basis symmetries (5) generates an adjoint representation (or interior automorphism) \( \text{Ad}(\exp(\varepsilon \Gamma_i)) \Gamma_j \) defined by Ref. 8,

\[ \text{Ad}(\exp(\varepsilon \Gamma_i)) \Gamma_j = \Gamma_j - \varepsilon[\Gamma_i, \Gamma_j] + \frac{1}{2} \varepsilon^2 [\Gamma_i, [\Gamma_i, \Gamma_j]] - \cdots. \]

Here, \([\Gamma_i, \Gamma_j]\) is the commutator given by

\[ [\Gamma_i, \Gamma_j] = \Gamma_i \Gamma_j - \Gamma_j \Gamma_i. \]

The commutator table of the Lie point symmetries of system (2) and the adjoint representations of the symmetry group of (2) on its Lie algebra are given in Tables I and II, respectively. Essentially these adjoint representations simply permute amongst “similar” one-dimensional subalgebras. Hence, they are used to identify similar one-dimensional subalgebras. However, before proceeding with the classification scheme one needs to identify invariants of the full adjoint action as these place restrictions on how far one can expect to “simplify” a given arbitrary element,

\[ \Gamma = \sum_{i=1}^{5} C_i \Gamma_i, \]

### TABLE I. Commutator table of the Lie algebra of system (2).

<table>
<thead>
<tr>
<th>([\Gamma_i, \Gamma_j])</th>
<th>(\Gamma_1)</th>
<th>(\Gamma_2)</th>
<th>(\Gamma_3)</th>
<th>(\Gamma_4)</th>
<th>(\Gamma_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Gamma_1)</td>
<td>0</td>
<td>0</td>
<td>(\Gamma_4)</td>
<td>(-2\Gamma_5)</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_2)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_3)</td>
<td>(-\Gamma_4)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_4)</td>
<td>(2\Gamma_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\Gamma_5)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
of the GZEs Lie algebra $\mathfrak{g}_5$. A real function $\eta : \mathfrak{g}_5 \to \mathbb{R}$ defined by $\eta(\Gamma) = \psi(C_1, C_2, C_3, C_4, C_5)$ for some other function $\psi$ is an invariant of the full adjoint action in Table II if for each nonzero $\Gamma \in \mathfrak{g}_5$,

$$\eta(\text{Ad}(\exp(e\Gamma_i)))\Gamma = \eta(\Gamma), \quad i = 1, \ldots, 5.$$  

Note that elements of $\mathfrak{g}_5$ can be represented by vectors $A = (C_1, \ldots, C_5) \in \mathbb{R}^5$ since each of them can be written in the form $\Gamma = \sum_{i=1}^{5} C_i \Gamma_i$ for some constants $C_1, \ldots, C_5$. Thus, the adjoint action can be viewed as (in fact is) a group of linear transformations of the vectors $(C_1, \ldots, C_5)$. The adjoint representation group is generated (via Lie equations) by the Lie algebra $\mathfrak{g}_5^\Lambda$ spanned by the symmetries

$$\Delta_i = c^k_{ij} e^j \frac{\partial}{\partial e^k}, \quad i = 1, \ldots, 5,$$

where $c^k_{ij}$ are the structure constants in Table I. Explicitly, one has

$$\Delta_1 \eta(\Gamma) = C_3 \frac{\partial \eta(\Gamma)}{\partial C_4} - 2C_4 \frac{\partial \eta(\Gamma)}{\partial C_5} = 0, \quad \Delta_2 \eta(\Gamma) = C_1 \frac{\partial \eta(\Gamma)}{\partial C_4} = 0, \quad \Delta_3 \eta(\Gamma) = 2C_1 \frac{\partial \eta(\Gamma)}{\partial C_5} = 0. \quad (7)$$

The invariants $\eta$ place a restriction on how far one can expect to simplify $\Gamma$ by the action of adjoint maps. For any other composition of adjoint maps there does not exist an $\alpha$ and $\beta$ such that some of the coefficients $c_1, c_2, c_3, c_4$, and $c_5$ in

$$X = \sum_{i=1}^{5} c_i \Gamma_i = \text{Ad}(\exp(\alpha \Gamma_i)) \circ \text{Ad}(\exp(\beta \Gamma_j)) \Gamma \quad i, j = 1, \ldots, 5, \quad (8)$$

are eliminated simultaneously.

Based on (6)-(8) and the classification of the one-dimensional subalgebras of the GZEs Lie algebra $\mathfrak{g}_5$, from Tables I and II one can obtain easily an optimal system of one-dimensional subalgebras given by $\{c\Gamma_1 + \Gamma_2, c\Gamma_1 + \Gamma_2 + \Gamma_3, c\Gamma_1 + \Gamma_2 + c_4\Gamma_4 + c_5\Gamma_5, c_1\Gamma_1 + \Gamma_2 + c_3\Gamma_3 + c_4\Gamma_4 + c_5\Gamma_5\}$, where $c_i = \frac{C_i}{C_5}$, $i = 1, 2, 3, 4, 5$.

The Lie algebra, spanned by infinitesimal generators (5), is the direct sum of $\Gamma_2$ and the nilpotent four-dimensional Lie algebra $\mathfrak{u}_{4,1}$ (see Ref. 29).

B. Some symmetry reductions and exact solutions

In this subsection we use the whole Lie algebra and the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions that yield solutions of (2) from solutions of corresponding systems of ordinary differential equations (ODEs). In order to obtain symmetry reductions and exact solutions, one has to solve the associated characteristic equations

$$\frac{dx}{\xi_1(x,t,u,v,F)} = \frac{dr}{\xi_2(x,t,u,v,F)} = \frac{du}{\eta_1(x,t,u,v,F)} = \frac{dv}{\eta_2(x,t,u,v,F)} = \frac{dF}{\eta_3(x,t,u,v,F)}. \quad (9)$$

We consider the following cases.
Case 1. \( C_3 = C_4 = C_5 = 0 \) and \( C_1 = c, \quad C_2 = 1 \).

Here, the solution of the corresponding characteristic system for the subalgebra \( c\Gamma_1 + \Gamma_2 \) a combination of the space-time translation symmetries of the symmetry field \( \Gamma \) gives rise to traveling wave group-invariant solutions of the form

\[
\begin{align*}
\xi & = x - ct \quad \text{is an invariant of the symmetry} \ c\Gamma_1 + \Gamma_2, \quad \text{and} \ c \quad \text{is an arbitrary constant. The functions} \ U, \ V, \ \text{and} \ F \ \text{satisfy}
\end{align*}
\]

\[
\begin{align*}
2FU + U'' - 2\beta U(U^2 + V^2) + cV' &= 0, \\
2FV - cU' - 2\beta V(U^2 + V^2) + V'' &= 0, \\
(c^2 - 1)F'' + 2(U^2 + V^2)'' &= 0.
\end{align*}
\] (11)

Integrating the third equation of (11) twice, we obtain

\[
F = \frac{2(U^2 + V^2)}{1 - c^2} + a\xi + b.
\] (12)

where \( a, b \) are two integration constants.

Then, inserting (12) into the first and second equations of (11), we get

\[
\begin{align*}
U'' + cV' &= -2U(a\xi + b), \\
cU' - V'' &= 2V(a\xi + b),
\end{align*}
\] (13)

where \( c = \pm \sqrt{1 - \frac{2}{\beta}} \). For (13), we consider two cases.

1. When \( ab \neq 0 \), we obtain the solution

\[
\begin{align*}
U &= \left[ c_1 \cos\left(\frac{1}{2}c\xi\right) + c_3 \sin\left(\frac{1}{2}c\xi\right)\right]Ai(\xi) + \left[ c_2 \cos\left(\frac{1}{2}c\xi\right) + c_4 \sin\left(\frac{1}{2}c\xi\right)\right]Bi(\xi), \\
V &= \left[ c_1 \sin\left(\frac{1}{2}c\xi\right) - c_3 \cos\left(\frac{1}{2}c\xi\right)\right]Ai(\xi) + \left[ c_2 \sin\left(\frac{1}{2}c\xi\right) - c_4 \cos\left(\frac{1}{2}c\xi\right)\right]Bi(\xi),
\end{align*}
\] (14)

where \( c_1, c_2, c_3, c_4 \) are arbitrary constants, \( \xi = -\frac{1}{8} \sqrt{\frac{2}{\beta}}(8a\xi + 8b + c^2) \), and \( Ai, Bi \) are the Airy wave functions of the first and second kind, respectively. Then, after substituting (14) into (12), we obtain

\[
\begin{align*}
F &= \beta \left[ c_1 Ai(\xi) \sin\left(\frac{c\xi}{2}\right) - c_3 Ai(\xi) \cos\left(\frac{c\xi}{2}\right) + c_2 Bi(\xi) \sin\left(\frac{c\xi}{2}\right) - c_4 Bi(\xi) \sin\left(\frac{c\xi}{2}\right)\right]^2 \\
&\quad + \left[ c_1 Ai(\xi) \cos\left(\frac{c\xi}{2}\right) + c_2 Bi(\xi) \cos\left(\frac{c\xi}{2}\right) + \sin\left(\frac{c\xi}{2}\right)(c_3 Ai(\xi) + c_4 Bi(\xi))\right]^2 + a\xi + b.
\end{align*}
\] (15)

The profile of solution (13)-(15) is given in Figs. 1 and 2.

![Evolution of travelling wave solution](image-url)

**FIG. 1.** Evolution of travelling wave solution (14) and (15) with parameters \( a = 5, b = 80, c_1 = c_2 = 1, c_3 = 2, c_4 = 3, \beta = 4 \).
FIG. 2. Evolution of travelling wave solution (14) and (15) at $x = 0$ with parameters $a = 5, b = 80, c_1 = c = 1, c_2 = 2, c_3 = 3, c_4 = 4, \beta = 50$.

(2) When $a = 0$ and $c^2 + 8b > 0$, we obtain the periodic solution

$$U = n_1 \cos(A\xi) + n_2 \sin(A\xi) + n_3 \cos(B\xi) + n_4 \sin(B\xi),$$

$$V = -n_1 \sin(A\xi) + n_2 \cos(A\xi) - n_3 \sin(B\xi) + n_4 \cos(B\xi),$$

where $A = -\frac{1}{2} + \frac{1}{2} \sqrt{c^2 + 8b}$, $B = -\frac{1}{2} - \frac{1}{2} \sqrt{c^2 + 8b}$, and $n_1, n_2, n_3, n_4$ are arbitrary constants. Then, after substituting (16) into (12), we get

$$F = \beta \left\{ (n_2 \sin(A\xi) + n_1 \cos(A\xi) + n_4 \sin(B\xi) + n_3 \cos(B\xi))^2 + (n_1 \sin(A\xi) - n_2 \cos(A\xi) + n_3 \sin(B\xi) - n_4 \cos(B\xi))^2 \right\} + b. \quad (17)$$

The profile of solution (16)-(17) is given in Figs. 3 and 4.

**Case 2.** $C_3 = C_4 = 0, C_1 = c$, and $C_2 = C_5 = 1$.

The subalgebra $c\Gamma_1 + \Gamma_2 + \Gamma_3$, which is a combination of the translation and rotation symmetries of the symmetry field $\Gamma$ gives rise to the group-invariant solution of the form

$$G^2(\xi) = u^2 + v^2,$$

$$u = G(\xi) \cos(x + K(\xi)),$$

$$v = G(\xi) \sin(x + K(\xi)),$$

$$F = F(\xi), \quad (18)$$

where $\xi = x - ct$ is an invariant of the symmetry $c\Gamma_1 + \Gamma_2 + \Gamma_3$, $c$ is a constant wave speed, and the similarity functions $G, K$, and $F$ satisfying the following similarity reduction equations:

FIG. 3. Evolution of travelling wave solution (16) and (17) with parameters $b = 80, n_1 = 4, n_2 = 3, c_3 = 2, n_4 = c = 1, \beta = 10$. 

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Three subcases arise as follows.

**Subcase I.** \( c \neq 1 \).

Integrating the second equation of (19) once after multiplication by \( G(\xi) \) and then the third equation of (19) twice, we have

\[
K'(\xi) = \frac{c - 2}{2} + \frac{\lambda_1}{2G^2},
\]

\[
F(\xi) = \frac{1}{c^2 - 1}(-G^2 + \lambda_2 + \lambda_3\xi),
\]

where \( \lambda_1, \lambda_2, \) and \( \lambda_3 \) are integration constants.

Substituting (20) into the first equation of (19), we get the nonlinear ODE

\[
2\left(\frac{1}{1 - c^2} - \beta\right)G^3 + \left(\frac{c^2}{4} - c + 2\lambda_2 + 2\lambda_3\xi\right)G + G'' - \frac{\lambda_1^2}{4G^3} = 0.
\]

We are unable to solve (21). In the special case \( \lambda_3 = 0 \), multiplying (21) by \( G'(\xi) \) and integrating once, we obtain

\[
4\left(\frac{1}{1 - c^2} - \beta\right)G^3 + [(c - 4)c + 8\lambda_2]G^2 + 4G'^2 + \frac{\lambda_1^2}{G^2} = 8k_1,
\]

where \( k_1 \) is the integration constant. From (22), we find the general solution

\[
\int \frac{2\sqrt{c^2 - 1}G}{\sqrt{8(c^2 - 1)k_1G^2 + c(c^2 - 1)(c^2 - 4c + 8\lambda_2)G^4 + 4[\beta(c^2 - 1) + 1]G^6 + (1 - c^2)\lambda_1^2}}\,dG = \xi + k_2,
\]

where \( k_2 \) is a second integration constant.

If \( c = \pm \sqrt{1 - \frac{1}{\beta}} \), (23) becomes

\[
\int \frac{dg}{\sqrt{Ag^2 + Bg + C}} = \xi + k_2,
\]

where \( g = G^2, A = \pm \sqrt{1 - \frac{1}{\beta}(1 - \frac{1}{\beta}) + 4\sqrt{1 - \frac{1}{\beta}(1 - \frac{1}{\beta}) + 8\lambda_2)}, B = 8k_1, \) and \( C = -\lambda_1^2 \).
Solving the left side integral of (24), we get
\[ \xi + k_2 = \int \frac{dg}{\sqrt{Ag^2 + Bg + C}} \]
\[ = \begin{cases} 
\frac{1}{\sqrt{A}} \ln |2Ag + B|, & A > 0, \\
\frac{1}{\sqrt{A}} \ln |2Ag + B|, & A > 0, B^2 = 4AC, 2Ag + B > 0, \\
\frac{1}{\sqrt{-A}} \arcsin\left(\frac{2Ag + B}{\sqrt{4AC - B^2}}\right), & A < 0, 4AC < B^2, \]
\[ \frac{1}{\sqrt{-A}} \ln |2Ag + B|, & A > 0, B^2 = 4AC, 2Ag + B < 0, \\
\frac{1}{\sqrt{-A}} \arcsin\left(\frac{2Ag + B}{\sqrt{4AC - B^2}}\right), & A < 0, 4AC < B^2, \]
\[ 2Ag + B < \sqrt{B^2 - 4AC}. \]
(25)

**Subcase II.** \(c = 1\).
Here, system (19) becomes
\[ G(2F - K^2 - K' - 1) + G'' - 2\beta G^3 = 0, \]
\[ G'(2K' + 1) + GK'' = 0, \]
(26)
which has the general solution
\[ G = \sqrt{\delta_1 \xi + \delta_2}, \]
\[ K = \frac{(\delta_2 + 2\delta_3) \log(\delta_1 \xi + \delta_2)}{2\delta_1} + \delta_4 - \frac{\xi}{2}, \]
\[ F = \frac{8\beta \delta_1^2 \xi^3 + \delta_1^2 (24\beta \delta_2 \xi^2 + 3\xi^2 + 1) + 6\delta_2 \delta_3 \xi (4\beta \delta_2 + 1) + 4(2\beta \delta_2^2 + \delta_2^2 + 3\delta_2 + \delta_3^2)}{8(\delta_1 \xi + \delta_2)^2}, \]
(27)
where \(\delta_1, \delta_2, \delta_3, \delta_4\) are arbitrary constants.

**Subcase III.** \(K(\xi) = A\xi = (\lambda - 1)x + \mu\), i.e., \(A = \lambda - 1, Ac = -\mu\), (18) becomes
\[ u = G(\xi) \cos(\lambda x + \mu), \]
\[ v = G(\xi) \sin(\lambda x + \mu), \]
(28)

Substitution of (28) into (2) results in the following system:
\[ (c - 2\lambda) \sin(\eta)G' + \cos(\eta)G(2F - \mu - \lambda^2) + \cos(\eta)G'' - 2\beta \cos(\eta)G^3 = 0, \]
\[ -(c - 2\lambda) \cos(\eta)G' + \sin(\eta)G(2F - \mu - \lambda^2) + \sin(\eta)G'' - 2\beta \sin(\eta)G^3 = 0, \]
\[ (c^2 - 1)F'' + 2GG'' + 2G^2 = 0, \]
(29)
with \(\eta = \lambda x + \mu\). Multiplying the first and second equations of (29) by \(\cos(\eta)\) and \(\sin(\eta)\), respectively, and then simplifying, we obtain the ODE
\[ G(2F - \mu - \lambda^2) + G'' - 2\beta G^3 = 0. \]
(30)

By integrating the third equation of (29) with respect to \(\xi\) twice, we obtain
\[ (c^2 - 1)F + G^2 = \lambda_1 \xi + \lambda_2, \]
(31)
where \(\lambda_1, \lambda_2\) are the resulting two integration constants.
In order to have a bounded solution, it follows that \( \lim_{\xi \to \infty} F = 0 \), \( \lim_{\xi \to \infty} F' = 0 \), \( \lim_{\xi \to \infty} G = 0 \). Accordingly from Eq. (31), we set \( \lambda_1 = \lambda_2 = 0 \), and thus

\[ F = \frac{G^2}{1 - c^2}, \quad (32) \]

In view of (32), Eq. (30) becomes

\[ 2(\beta + \frac{1}{c^2 - 1})G^3 - G'' + (\mu + \lambda^2) G = 0. \quad (33) \]

In Refs. 20 and 21, the authors found a particular solution of (33) by applying the semi-inverse method. Now, in order to obtain the general solution of (33), making an integrating factor \( G' \), and integrating once, Eq. (33) becomes

\[ (\beta + \frac{1}{c^2 - 1})G^4 - G'' + (\mu + \lambda^2) G^2 = 2q, \quad (34) \]

where \( q \) is integration constant. From (34), we get the general solution

\[ \int \frac{dG}{\sqrt{NG^4 + MG^2 - 2q}} = \xi, \quad (35) \]

where \( N = \beta + \frac{1}{c^2 - 1} \) and \( M = \mu + \lambda^2 \). Solving the left hand integral of (35), we get the following two results.

1. When \( q \neq 0 \), we obtain

\[ \xi = \int \frac{dG}{\sqrt{NG^4 + MG^2 - 2q}} = \frac{\sqrt{2}}{D_1} F(\frac{D_1}{2\sqrt{q}} G, D_2), \quad (36) \]

where \( D_1 = \sqrt{M^2 - 8Nq - M} \), \( D_2 = \frac{1}{2} \sqrt{\frac{M^2 + M\sqrt{M^2 - 8Nq - 4Nq}}{Nq}} \), and \( F \) is the incomplete elliptic integral of the first kind.

2. When \( q = 0 \), we obtain

\[ \xi = \int \frac{dG}{\sqrt{NG^4 + MG^2 - 2q}} = \pm \frac{1}{\sqrt{M}} \ln | \sqrt{\frac{M}{N}} + \sqrt{\frac{G^2 + M}{N}} |. \quad (37) \]

**Case 3.** \( C_i, \ i = 1, 2, \ldots, 5 \) are arbitrary constants.

For the most general generator \( \Gamma = c_1 \Gamma_1 + \Gamma_2 + c_3 \Gamma_3 + c_4 \Gamma_4 + c_5 \Gamma_5 \), \( c_i = \frac{C_i}{C_2}, \ i = 1, 2, 3, 4, 5 \), the corresponding invariant solutions of (2) are given by

\[ u(x,t) = -A_1(\xi) \sin(\frac{1}{3} c_3 t^3 + c_4 t^2 - c_5 t) + A_2(\xi) \cos(\frac{1}{3} c_3 t^3 + c_4 t^2 - c_5 t), \]

\[ v(x,t) = A_2(\xi) \sin(\frac{1}{3} c_3 t^3 + c_4 t^2 - c_5 t) + A_1(\xi) \cos(\frac{1}{3} c_3 t^3 + c_4 t^2 - c_5 t), \]

\[ F(x,t) = \frac{1}{2} c_3 t^2 + c_4 t + H(\xi), \quad (38) \]

where \( A_1(\xi), A_2(\xi), \) and \( H(\xi) \) are functions of the similarity variable \( \xi = x - c_1 t \) that satisfy the nonlinear system of ODEs

\[ -2\beta A_3^2 + A_2(-2\beta A_1^2 + c_5 + 2H) + c_1 A_1' + A_2'' = 0, \]

\[ 2\beta A_3^2 - A_1(-2\beta A_1^2 + c_5 + 2H) + c_1 A_2' - A_3'' = 0, \]

\[ (A_1^2 + A_2^2)'' + (c_1^2 - 1)H'' + c_3 = 0. \quad (39) \]

Integrating the third equation of (39) with respect to \( \xi \) twice, and letting \( c_1 = \sqrt{1 - \frac{1}{\beta}} \), \( (\beta < 0 \text{ or } \beta \geq 1) \), yields

\[ H = \beta(A_1^2 + A_2^2) + c_3 \beta \xi^2 + k \xi + \mu, \quad (40) \]
where \( \kappa, \mu \) are integration constants. We substitute (40) into the first and second equations of (39) to obtain a linear ordinary differential system,

\[
[c_5 + 2(\mu + \kappa \xi + c_3 \beta \xi^2)]A_2 + \sqrt{1 - \frac{1}{\beta}} A_1' + A_2'' = 0,
\]

\[
[c_5 + 2(\mu + \kappa \xi + c_3 \beta \xi^2)]A_1 - \sqrt{1 - \frac{1}{\beta}} A_2' + A_1'' = 0.
\]

(41)

In order to solve (41), we consider the following subcases:

**Subcase I.** \( c_3 = 0 \), i.e., invariant solutions corresponding to the subalgebra \( \Gamma = c_1 \Gamma_1 + \Gamma_2 + c_4 \Gamma_4 + c_5 \Gamma_5 \).

(I-1) When \( \kappa = 0 \), we get the ODE system

\[
(c_5 + 2\mu)A_2 + \sqrt{1 - \frac{1}{\beta}} A_1' + A_2'' = 0,
\]

\[
(c_5 + 2\mu)A_1 - \sqrt{1 - \frac{1}{\beta}} A_2' + A_1'' = 0,
\]

(42)

whose general solution is given by

\[
A_1 = b_1 \cos(\delta \xi) + b_2 \sin(\delta \xi) + b_3 \cos(\sigma \xi) + b_4 \sin(\sigma \xi),
\]

\[
A_2 = -b_1 \sin(\delta \xi) + b_2 \cos(\delta \xi) - b_3 \sin(\sigma \xi) + b_4 \cos(\sigma \xi),
\]

(43)

with \( \Delta = \frac{\beta-1}{\beta} + 4(c_5 + 2\mu) > 0 \), \( \delta = \frac{1}{2}(\sqrt{\frac{\beta-1}{\beta} + \sqrt{\Delta}}) \), and \( \sigma = \frac{1}{2}(\sqrt{\frac{\beta-1}{\beta} - \sqrt{\Delta}}) \), and \( b_i, i = 1, 2, 3, 4 \) are four arbitrary constants.

Thus, from (38), (40), (43), and the transformation \( E = u + iv \), we obtain the following exact solution of the GZEs (1):

\[
E(x, t) = [b_1 \cos(\delta \xi) + b_2 \sin(\delta \xi) + b_3 \cos(\sigma \xi) + b_4 \sin(\sigma \xi)][-\sin(\xi) + i \cos(\xi)]
\]

\[
+[-b_1 \sin(\delta \xi) + b_2 \cos(\delta \xi) - b_3 \sin(\sigma \xi) + b_4 \cos(\sigma \xi)][\cos(\xi) + i \sin(\xi)],
\]

\[
F(x, t) = \beta \{[b_1 \cos(\delta \xi) + b_2 \sin(\delta \xi) + b_3 \cos(\sigma \xi) + b_4 \sin(\sigma \xi)]^2
\]

\[
+[-b_1 \sin(\delta \xi) + b_2 \cos(\delta \xi) - b_3 \sin(\sigma \xi) + b_4 \cos(\sigma \xi)]^2\} + c_4 t,
\]

(44)

where \( \xi = c_4 t^2 - c_5 t \).

(I-2) When \( \beta = 1 \), we get the ODE system

\[
[c_5 + 2(\mu + \kappa \xi)]A_i + A_i'' = 0, \quad i = 1, 2,
\]

(45)

whose solution is given by

\[
A_1 = A_2 = \sqrt{\eta}[\alpha_1 J_{1/\beta}(\frac{2}{3} \sqrt{2\kappa \eta^{3/2}}) + \alpha_2 Y_{1/\beta}(\frac{2}{3} \sqrt{2\kappa \eta^{3/2}})],
\]

(46)

with \( \eta = \xi + \frac{c_5^2 \mu}{\kappa} \); \( J, Y \) are Bessel functions, and \( \alpha_1, \alpha_2 \) are two arbitrary constants.

Consequently, from (38), (40), (46), and the transformation \( E = u + iv \), we obtain the exact solution of the GZEs (1) given by

\[
E(x, t) = \sqrt{\eta}[\alpha_1 J_{1/\beta}(\frac{2}{3} \sqrt{2\kappa \eta^{3/2}}) + \alpha_2 Y_{1/\beta}(\frac{2}{3} \sqrt{2\kappa \eta^{3/2}})][(1 + i) \cos(\xi) + (1 - i) \sin(\xi)],
\]

\[
F(x, t) = 2\eta[\alpha_1 J_{1/\beta}(\frac{2}{3} \sqrt{2\kappa \eta^{3/2}}) + \alpha_2 Y_{1/\beta}(\frac{2}{3} \sqrt{2\kappa \eta^{3/2}})]^2 + \kappa \xi + \mu + c_4 t,
\]

(47)

where \( \xi = c_4 t^2 - c_5 t \).
Subcase II. $c_3 \neq 0$, i.e., under the whole algebra $\Gamma = c_1 \Gamma_1 + c_2 \Gamma_2 + c_3 \Gamma_3 + c_4 \Gamma_4 + c_5 \Gamma_5$.

(II-1) When $\kappa = 0$ and $c_3 = -\theta (\theta > 0)$, we get the ODE system

\[
[c_5 + 2(\mu - \theta \xi^2)]A_2 + \sqrt{1 - \frac{1}{\beta}} A'_2 + A''_2 = 0,
\]

\[
[c_5 + 2(\mu - \theta \xi^2)]A_1 - \sqrt{1 - \frac{1}{\beta}} A'_1 + A''_1 = 0,
\]

whose general solution is given by

\[
A_1 = \frac{1}{\sqrt{\xi}} [\lambda_1 M_{k,m}(\phi) \cos(\varphi) + \lambda_2 M_{k,m}(\phi) \sin(\varphi) + \lambda_3 W_{k,m}(\phi) \cos(\varphi) + \lambda_4 W_{k,m}(\phi) \sin(\varphi)],
\]

\[
A_2 = \frac{1}{\sqrt{\xi}} [-\lambda_1 M_{k,m}(\phi) \sin(\varphi) + \lambda_2 M_{k,m}(\phi) \cos(\varphi) - \lambda_3 W_{k,m}(\phi) \sin(\varphi) + \lambda_4 W_{k,m}(\phi) \cos(\varphi)],
\]

where $M_{k,m}(\phi), W_{k,m}(\phi)$ are the Whittaker functions, $\phi = \sqrt{2\beta\theta\xi^2}, \varphi = \frac{1}{2} \sqrt{1 - \frac{1}{\beta}} \xi, k = \frac{\beta - 1 + 8\theta \mu + 4\theta \xi \sqrt{\xi}}{32 \beta^{1/2} \sqrt{\theta}}, m = \frac{3}{4}$, and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are arbitrary constants.

Thus, from (38), (40), (49), and the transformation $E = u + iv$, we obtain the exact solution of the GZE (1) given by

\[
E(x,t) = [-A_1 \sin(\psi) + A_2 \cos(\psi)] + i[A_2 \sin(\psi) + A_1 \cos(\psi)],
\]

\[
F(x,t) = \beta(A_1^2 + A_2^2) + c_3 \xi^2 + \kappa \xi + \mu + \frac{1}{2} c_4 t^2 + c_4 t,
\]

where $\psi = \frac{1}{4} c_3 t^3 + c_4 t^2 - c_4 t$, and $A_1, A_2$ are given by (49).

(II-2) When $\beta = 1$ and $c_3 = -\theta (\theta > 0)$, we get an ODE system

\[
[c_5 + 2(\mu + \kappa \xi - \theta \xi^2)]A_i + A'_i = 0, \quad i = 1, 2,
\]

which has the solution of the form

\[
A_1 = A_2 = e^{\frac{\sqrt{2} \xi (6\mu - \kappa)}{2 \xi}} [\mu_1 F(\frac{3}{4} - \Omega, \frac{3}{4} - \Omega, \Omega, \frac{3}{4}, \omega) + \mu_2 (2\theta \xi - \kappa) F(\frac{3}{4} - \Omega, \frac{3}{4}, \Omega, \frac{3}{4}, \omega)],
\]

with $\Omega = \sqrt{2(2\mu \xi + 4\theta \mu + \kappa)}$, $\omega = \sqrt{2 \theta \xi (6\mu - \kappa)}$, $F$ is the generalized hypergeometric function, and $\mu_1, \mu_2$ are two arbitrary constants.

Thus, from (38), (40), (52), and the transformation $E = u + iv$, we obtain the exact solution of the GZE (1) given by

\[
E(x,t) = e^{\frac{\sqrt{2} \xi (6\mu - \kappa)}{2 \xi}} [\mu_1 F(\frac{3}{4} - \Omega, \frac{3}{4} - \Omega, \Omega, \frac{3}{4}, \omega) + \mu_2 (2\theta \xi - \kappa) F(\frac{3}{4} - \Omega, \frac{3}{4}, \Omega, \frac{3}{4}, \omega)]
\]

\[
x[1 + i \cos(\psi) + (1 - i) \sin(\psi)],
\]

\[
F(x,t) = e^{\frac{\sqrt{2} \xi (6\mu - \kappa)}{2 \xi}} [\mu_1 F(\frac{3}{4} - \Omega, \frac{3}{4} - \Omega, \Omega, \frac{3}{4}, \omega) + \mu_2 (2\theta \xi - \kappa) F(\frac{3}{4} - \Omega, \frac{3}{4}, \Omega, \frac{3}{4}, \omega)]
\]

\[
-\theta \xi^2 + \kappa \xi + \mu - \frac{1}{2} \theta t^2 + c_4 t,
\]

where $\psi = -\frac{1}{4} \theta t^3 + c_4 t^2 - c_4 t$.

III. CONSERVATION LAWS

Let $x = (x_1, x_2, \ldots, x_n)$ be $n$ independent variables and $u = (u^1, u^2, \ldots, u^m)$ be $m$ dependent variables. Consider a system of $r$ PDEs of $k$th-order given by

\[
P_u[u] = P_u(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) = 0, \quad \alpha = 1, 2, \ldots, r,
\]
where \( u_{(1)} = \{u^\alpha_i\}, u_{(2)} = \{u^\alpha_{ij}\}, \ldots \), and \( u^\alpha_i = \frac{\partial u^\alpha}{\partial x_i}, u^\alpha_{ij} = \frac{\partial^2 u^\alpha}{\partial x_i \partial x_j}, \ldots \). We let \( U = (U^1, U^2, \ldots, U^N) \) denote arbitrary functions of the independent variables \( x \) and denote partial derivatives \( \partial / \partial x_i \) by subscripts \( i \), i.e., \( U^\alpha_i = \partial U^\alpha / \partial x_i, U^\alpha_{ij} = \partial^2 U^\alpha / \partial x_i \partial x_j, \) etc.

1. The total derivative operators \( D_i \) with respect to \( x_i \) are

\[
D_i = \frac{\partial}{\partial x_i} + u^\alpha_i \frac{\partial}{\partial u^\alpha_i} + u^\alpha_{ij} \frac{\partial}{\partial u^\alpha_{ij}} + u^\alpha_{ijk} \frac{\partial}{\partial u^\alpha_{ijk}} + \cdots ,
\]

where \( i, j, k, \ldots = 1, 2, \ldots, n \) and \( \alpha = 1, 2, \ldots, m \).

2. Multipliers for PDE system (54) are a set of functions \( \{\Lambda^\alpha[U]\} \) satisfying

\[
\Lambda^\alpha[U]P_\alpha[U] = D_iT^i[U],
\]

for some functions \( \{T^i[U]\} \).

If \( U^\alpha = \eta^\alpha(x) \) is a solution of PDE system (54), then from (56) we obtain the conservation law

\[
D_iT^i[u] = 0
\]

of system (54), and for each \( i, T^i[u] \) is a flux.

3. The standard Euler operators with respect to the differentiable function \( U^i \) and its derivatives \( U^i_1, U^i_1U^i_2, \ldots \) are the operators defined by

\[
E_{U^i} = \frac{\partial}{\partial U^i} - D_1 \frac{\partial}{\partial U^i_1} + \cdots + (-1)^sD_1 \cdots D_s \frac{\partial}{\partial U^i_1 \cdots i_s} + \cdots ,
\]

for each \( j = 1, 2, \ldots, m \).

\( \{\Lambda^\alpha[U]\} \) yields a set of multipliers for a conservation law of system (54) if and only if each Euler operator (58) annihilates the left-hand side of (56), i.e.,

\[
E_{U^i}(\Lambda^\alpha[U]P_\alpha[U]) \equiv 0, \ j = 1, \ldots, n,
\]

for arbitrary \( U, U_i, U_{ij}, \ldots \), etc.

From determining Equation (59) for multipliers, we obtain the zeroth-order multipliers (with the aid of GeM\(^3\)) \( \Lambda^1(x_1, x_2, U^1, U^2, U^3), \Lambda^2(x_1, x_2, U^1, U^2, U^3) \) and \( \Lambda^3(x_1, x_2, U^1, U^2, U^3) \) for GZE (2) which are given by

\[
\begin{align*}
\Lambda^1 &= \frac{1}{2}(\gamma_1 t^2 + \gamma_2 x + \gamma_3)V, \\
\Lambda^2 &= -(\gamma_1 t^2 + \gamma_2 x + \gamma_3)U, \\
\Lambda^3 &= -\frac{1}{12}\gamma_1 t^3 - \frac{1}{4}\gamma_2 t^2 + \frac{1}{12}t(-3\gamma_1 x^2 + 12\gamma_6 x + 12\gamma_4) - \frac{1}{4}\gamma_2 x^2 + \gamma_7 x + \gamma_5,
\end{align*}
\]

where \( \gamma_i, i = 1, 2, \ldots, 7 \) are arbitrary constants and \( t = x_1, x = x_2, U = U^1, V = U^2 \).

Then, from (56) and (60), we have the following seven conserved vectors of (2) satisfying

\[
\frac{\partial}{\partial \bar{t}} T^i_\alpha[u, v, F] + \frac{\partial}{\partial x} T^i_\alpha[u, v, F] = 0, i = 1, 2, \ldots, 7
\]
\begin{align}
(T_1^+, T_2^+) &= \left(-\frac{1}{12}t^3 + \frac{1}{4}t x^2 F_x + \frac{1}{4} t^2 + \frac{1}{4} (u^2 + v^2),
(1 + \frac{1}{4} t x^2) F_t - \frac{1}{2} t x F - \left(\frac{1}{6} + \frac{1}{2} t x^2\right)(u u_x + v v_x) + \frac{1}{2} t x (u^2 + v^2),
(T_2^+, T_3^+) &= \left[-\frac{1}{4} (t^2 + x^2) F + \frac{1}{2} t F - \frac{1}{2} t (u^2 + v^2),
(1 + \frac{1}{4} t x^2) F_x - \frac{1}{2} x F - \frac{1}{2} (t^2 + x^2)(u u_x + v v_x) + t (u u_x - v v_x) + \frac{1}{2} x (u^2 + v^2),
(T_3^+, T_4^+) &= \left[-\frac{1}{2} (u^2 + v^2), v u_x - u v_x],
(T_4^+, T_5^+) &= \left[t F_t - F, t(2(u u_x + v v_x) - F_x),
(T_5^+, T_6^+) &= \left[F_t, 2(u u_x + v v_x) - F_x],
(T_6^+, T_7^+) &= \left[x(t F_t - F), t(-x F_x + F + 2x(u u_x + v v_x) - u^2 - v^2),
(T_7^+, T_8^+) &= \left[x F_t, -x F_x + F + 2x(u u_x + v v_x) - u^2 - v^2].
\end{align}

**Remark.** One can show that there are no higher-order (≥1) multipliers for conservation laws of Eqs. (2) by applying the direct method. Since there are seven potential (nonlocal) variables resulting from conservation laws (61) of Eqs. (2), one could obtain a tree of up to 127 nonlocally related PDE systems. Nonlocal symmetries of Eqs. (2) might arise from some of these nonlocally related PDE systems through applying Lie’s algorithm to each nonlocally related system in the tree.

### IV. CONCLUSIONS

In this paper, we used Lie symmetry analysis to obtain reductions and new exact solutions of the generalized Zakharov equations through optimal systems of one-dimensional Lie subalgebras. The exact solutions contained some new solutions, including solutions involving Airy wave functions, Bessel functions, Whittaker functions, or generalized hypergeometric functions. The correctness of the solutions has been verified by substituting them back in to GZEs. Second, the conservation laws of the GZEs have been constructed by using the direct (multiplier) method. Furthermore these solutions can be used as benchmarks against numerical solutions. It is worth mentioning that the current study for GZEs can be further pursued from the point of view of Hamiltonian structures and numerical simulations.

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